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GEOMETRY OF MANIFOLDS OF MAPS

HALLDOR I. ELIASSON

Introduction

The main purpose of this paper is to lift differential geometric objects from two manifolds N and M to a Banach manifold $\mathfrak{S}(N, M)$ of maps from N to M. To give an explicit construction of such objects, it seems to me, is fundamental for analysis on these manifolds. Moreover, we are able to prove existence where conventional methods fail, because of the lack of smooth partitions of unity on many Banach manifolds of maps as e.g. $C^k(N, M)$.

The general setting for manifolds of maps is as follows (§4): N is a compact Hausdorff space of class C^r with a countable basis $0 \leq r \leq \infty$ (C^r-manifold for $r \geq 1$), $\mathfrak{A}$ is a full subcategory of the category $\mathfrak{B}$ of Banach spaces closed under the operations of taking direct sums and continuous linear maps. $VB(N, \mathfrak{A})$ is the category of vector bundles of class C^r over N with fibres in $\mathfrak{A}$. $\mathfrak{S}$ is a section functor, which is a manifold model, t.i. $\mathfrak{S}$ is a covariant functor from $VB(N, \mathfrak{A})$ into $\mathfrak{B}$ s.t. $\mathfrak{S}(E)$ is a Banach space of sections in E for any $E \in VB(N, \mathfrak{A})$ and as a manifold model $\mathfrak{S}$ has the following three properties: we have continuous linear inclusions $\mathfrak{S}(E) \subset C^0(E)$ and $\mathfrak{S}(L(E, F)) \subset L(\mathfrak{S}(E), \mathfrak{S}(F))$ and the map of section spaces induced by C^r-fibre maps via $\mathfrak{S}$ is continuous. We then prove differentiability and give an explicit formula for the derivatives (Lemma 4.1). This is the fundamental lemma for most of the following constructions.

Then (§5) M is a Banach manifold of class C^{r+s}, $s \geq 3$, admitting a connection of class C^{r+s-2} and modeled on $\mathfrak{A}$. we then construct a C^{s-2}-Banach manifold $\mathfrak{S}(N, M)$ of maps. We follow the idea in [3] to use the exponential map for the construction of a chart in a neighborhood of a C^r-map $h: N \to M$ using $\mathfrak{S}(h^*TM)$ as a model.

This axiomatic setting is slightly more general than in Palais [5], where N, M are finite dimensional of class C^∞, so we have included more of the known examples due to Eells [3] as e.g. $C^0(N, M)$, where N is some space as above ($r = 0$). Moreover it is important to allow M to be infinite dimensional, for $\mathfrak{S} = C^k$, this has already been worked out, Abraham [1]. Furthermore, the axioms for a manifold model are slightly more general. The fundamental difference, however, lies in the fact that we start globally on

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vector bundles and use only differential geometric objects and structures from N and M to carry through the constructions.

We then proceed to the proof, that the tangent bundle of $\mathfrak{S}(N, M)$ can be identified with $\mathfrak{S}(N, TM)$ in a natural manner and that a map $\theta: M \to M'$ of class C^{r+s-2} induces by composition a map $\mathfrak{S}(\theta): \mathfrak{S}(N, M) \to \mathfrak{S}(N, M')$ of class C^{s-2} and $T\mathfrak{S}(\theta) = \mathfrak{S}(T\theta)$. Our main result is Theorem 4.5, where we prove that a C^{r+s-2}-connection on M induces a canonical connection of class C^{r+s-4} on $\mathfrak{S}(N, M)$, such that if exp denotes the exponential map for M, then $\mathfrak{S}(\exp)$ is the exponential map for $\mathfrak{S}(N, M)$. This implies that if N, M are of class C^∞ and M admits a C^∞-connetion, then the C-manifold $C^k(N, M)$ admits a connection of class C^∞. We then have the iterated manifolds of maps $C^l(N, C^k(N, M))$ etc. Note that here a C^∞ connection cannot be constructed using C^∞ partitions of unity, as they do not exist (see [4] for references).

In §6 we construct vector bundles over $\mathfrak{S}(N, M)$ and bundle maps between those. Let $\lambda: \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ be a functor of class C^∞ and λ_N the induced operation on $VB(N, \mathfrak{B})$, $\mathfrak{T}$ a section functor such that we have a continuous linear inclusion $\mathfrak{S}(L(E, F)) \subset L(\mathfrak{T}(E), \mathfrak{T}(F))$. Let $E \in VB(N, \mathfrak{A})$ be given. We then prove that the domain of definition for $\mathfrak{T}$ can be extended as to define $\mathfrak{T}(\lambda_N(E, f^*TM))$ for all $f \in \mathfrak{S}(N, M)$ and the union of these Banach spaces is a vector bundle $\mathfrak{T}(\lambda_N(E, \mathfrak{S}(N, M)^*TM))$ over $\mathfrak{S}(N,M)$ of class C^{s-3}. In particular, we have the bundles $\mathfrak{S}(\lambda_N(\mathfrak{S}(N, M)^*TM))$, which are just lifts of the tensor bundles over M with obvious lifts of sections (for $\lambda = \mathrm{id}$ we have the tangent bundle of $\mathfrak{S}(N, M)$). The importance of this lies also in the fact that we usually have to consider weaker variations of mappings than those obtainable through manifold model sections.

In Theorems 6.2-6.4 we give sufficient conditions, that the tangent extends to a section ∂ in $\mathfrak{T}(L(TN, \mathfrak{S}(N, M)^*TM))$ of class $C^{s-3}(r \geq 1)$ and that a covariant differentiation in $TN(r \geq 2)$ and TM induces sections in

$$L(\mathfrak{T}_1(L^k(TN, \mathfrak{S}(N, M)^*TM)), \mathfrak{T}_2(L^{k+1}(TN, \mathfrak{S}(N, M)^*TM)))$$

$k \geq 0$ of class C^{s-4}. This has immediate applications to the chain $C^0, C^1, \cdots$ of manifold models. However, the most important applications will be to the Sobolev chain $H^0, H^1, \cdots, H^k$ $(2k > \dim N)$. I do not give any applications to $H^k(= L_k^2)$ here, but I intend to prove in a later paper that the k-th order energy function $E_k: H^k(N, M) \to R$, N and M compact without boundary, satisfies condition C of Palais and Smale (see [4] for references).

The above bundles and bundle maps form a natural setting for the study of partial differential equations for maps $N \to M$.

The first three sections are of a preparatory nature. In §1 we introduce connection for vector bundles via the connection map and compute canonical connections for the associated bundles of direct sums and linear maps. In §2 we define the covariant derivative of a section in $E \to N$ as a section in

$L(TN, E)$ and so on for higher derivatives using the induced connection. We furthermore observe that this covariant differentiation has properties similar to the usual differentiation in Banach spaces.

In §3 starting from a manifold M with connection, we define a certain induced connection for the total space TM of the tangent bundle and investigate the properties of the corresponding exponential map, which is to be used to obtain canonical charts for $\mathfrak{S}(N, TM)$. This connection may be characterized by the fact that the geodesics in TM are exactly the Jacobi fields along geodesics in M. We furthermore analyze the two first (covariant) derivatives of the exponential map for M, as we will need those for the construction of canonical local trivializations of our vector bundles over $\mathfrak{S}(N, M)$. The general reference here is [2] and [6]. The Jacobi connection mentioned above has also been constructed by J. Vilms [7].

1. Connection in vector bundles

Let M be a Banach manifold of class C^k, $k \geq 1$, with boundary ∂M. A chart for M is given by a diffeomorphism $\phi : U \to \phi U$ of class C^k, where U is open in M and ϕU is open in $\mathbf{M}_\lambda^+ = \{x \in \mathbf{M} : \lambda \cdot x \geq 0\}$, $\mathbf{M}$ is a Banach space and λ a functional on $\mathbf{M}$. $\partial U \subset \partial M$ is then given as the inverse image of $\partial\phi U = \phi U \cap \mathbf{M}_\lambda^0$, $\mathbf{M}_\lambda^0 = \{x \in \mathbf{M}: \lambda \cdot x = 0\}$ (see [6]). The restriction $\phi | \partial U$ gives a chart of class C^k for the manifold ∂M. The tangent of ϕ, $T\phi : TU \to \phi U \times \mathbf{M}$ gives a chart for the total space TM of the tangent bundle $\tau : TM \to M$. We have a C^{k-1} section l in $L(TM | \partial M, R)$ given locally by the functionals, such that $\ker l = T\partial M$ and a vector $v \in T_pM$, $p \in \partial M$, is tangent to M iff $l(p) \cdot v \geq 0$, in the sense that we have a curve c: $[0, 1] \to M$ of class C^1 with $\partial_c(0) = v$.

Let π: $E \to M$ be a vector bundle of class C^r, $1 \leq r \leq k$. A local trivialization for π is given by a bundle equivalence:

$$\begin{array}{lccc} \Phi : & \pi^{-1}(U) & \to & \phi U \times \mathbf{E} \\ & \downarrow & & \downarrow \\ \phi : & U & \to & \phi U \end{array}$$

where ϕ is a chart for M and $\mathbf{E}$ a Banach space. With $\xi \in E_p = \pi^{-1}(p)$ we have $\Phi(\xi) = (\phi(p), \Phi_p(\xi))$, where $\Phi_p : E_p \to \mathbf{E}$ is a toplinear isomorphism. Given another local trivialization by (V, ψ, Ψ), we will define the transition map by :

$$G_{\psi\phi} : \phi(U \cap V) \to L(\mathbf{E}, \mathbf{E})\,;\; G_{\psi\phi}(x) = \Psi_p \circ \Phi_p^{-1},$$

with $x = \phi(p)$. $G_{\psi\phi}$ is then of class C^r. E is a Banach manifold of class C^r (with boundary) so the tangent bundle $\pi_1 : TE \to E$ is of class C^{r-1}. The tangent of π is a bundle map $T\pi$: $TE \to TM$ of class C^{r-1} and we see immediately from the local formulas

$$\Psi \circ \Phi^{-1}(x, \xi) = (\psi \circ \phi^{-1}(x), G_{\psi\phi}(x) \cdot \xi),$$
$$T(\Psi \circ \Phi^{-1})(x, \xi, y, \eta) = (\psi \circ \phi^{-1}(x), G_{\psi\phi}(x) \cdot \xi, D(\psi \circ \phi^{-1})(x) \cdot y, DG_{\psi\phi}(x) \cdot (y, \xi) + G_{\psi\phi}(x) \cdot \eta),$$

that we can introduce vector space structure in the fibres of $T\pi$ (locally $T\pi$ maps (x, ξ, y, η) to (x, y)), to give $T\pi : TE \to TM$ a vector bundle structure of class C^{r-1}.

Definition. A connection map K for the bundle E is a map $K : TE \to E$, such that for any local trivialization (U, ϕ, Φ) of $\pi : E \to M$, there is a map $\Gamma_\phi : \phi U \to L(M, E; E)$ of class C^{r-1}, which gives the local representative of K, $K_\phi = \Phi \circ K \circ T\Phi^{-1}$ by the formula:

$$K_\phi(x, \xi, y, \eta) = (x, \eta + \Gamma_\phi(x) \cdot (y, \xi)).$$

It follows of course that K is of class C^{r-1}, but not conversely. We call Γ_ϕ the local connector for K and we will sometimes drop the suffix ϕ, if the local trivialization is fixed. In the case $E = TM$, $r = k - 1$, we have $G_{\psi\phi} = D(\psi \circ \phi^{-1})$ and the local connector corresponds to the classical Christoffel symbols, t.i. in the finite dimensional case we have in coordinates:

$$[\Gamma(x) \cdot (y, z)]^i = \Sigma \Gamma^i_{jk}(x) y^j z^k.$$

Using the formula for $T(\Psi \circ \Phi^{-1})$ above we get the following transformation formula for the local connector:

$$\Gamma_\phi(x) \cdot (y, \xi) = G_{\phi\psi}(\psi \circ \phi^{-1}(x)) \cdot [DG_{\psi\phi}(x) \cdot (y, \xi) + \Gamma_\psi(\psi \circ \phi^{-1}(x)) \cdot (D(\psi \circ \phi^{-1})(x) \cdot y, G_{\psi\phi}(x) \cdot \xi)].$$

Thus the required properties of the local connector are invariant under change of trivialization. It follows furthermore, that if M admits partition of unity of class C^{r-1}, then there exists a connection map for π.

We have two subbundles ker $T\pi$ and ker K of π_1, as $T\pi$ and K are both C^{r-1} surjective bundle maps with splitting kernels (see [6]). We have moreover an isomorphism ker $T\pi \oplus$ ker $K \cong \pi_1$ by $(A, B) \to A + B$. For this splitting of π_1 and later for covariant differentiation of sections we need our assumption on the "strong" differentiability of the connection, t.i. we are not satisfied with the weaker assumption that K or equivalently $(x, y, \xi) \to \Gamma(x) \cdot (y, \xi)$ is differentiable. A connection wtth this "weaker" differentiability is equivalent to a spray as defined in Lang [6] and suffices to introduce geodesics and the exponential map. Note that the map $(\pi_1, T\pi, K) : TE \to E \oplus TM \oplus E$ is a C^{r-1} diffeomorphism.

Let $\alpha : [0, 1] \to E$ be a C^1-curve in E. We denote by e the basis section of $TR = R \times R$, $e(t) = (t, 1)$. Then $\partial\alpha = T\alpha \circ e$ is the tangent field of α and the covariant derivative of α is defined by $\nabla\alpha = K \circ \partial\alpha$. We call α parallel, if $\nabla\alpha = 0$. If α is given in a local trivialization by $(\boldsymbol{c}, \boldsymbol{a})$, then $\nabla\alpha$ is given by

$$(\boldsymbol{c}, \boldsymbol{a}' + (\Gamma \circ \boldsymbol{c}) \cdot (\boldsymbol{c}', \boldsymbol{a})) .$$

It follows that if $c : [0, 1] \to M$ is a given C^1-curve and $\xi \in E_{c(0)}$, then there is a unique parallel curve α in E with $\pi \circ \alpha = c$ (or parallel field along c) and $\alpha(0) = \xi$. In the case $E = TM$, $\nabla\partial c = 0$ is the equation for a geodesic in M and there is a unique local solution with $\partial c(0) = v$, for a given v tangent to M.

We will now show that the category of vector bundles over M admitting a connection is closed under the operations of taking direct sums and linear maps.

Lemma 1.1. *Let $\pi : E \to M$ and $\rho : F \to M$ be vector bundles over M, of class C^r, then there is a natural bundle equivalence $T(\pi \oplus \rho) \cong T\pi \oplus T\rho$, given locally by an identity.*

Proof. Let $(A, B) \in TE \oplus_{TM} TF$, $T\pi(A) = T\rho(B) = v$. Let $c(t)$ be a C^1-curve in M with $\partial c(0) = v$ and α, β any fields along c in E, F with $\partial\alpha(0) = A$, $\partial\beta(0) = B$. Define $\gamma(t) = (\alpha(t), \beta(t))$ and let $(A, B) \cong \partial\gamma(0)$. This is easily seen to be an identity in a local trivialization using tangent charts.

Proposition 1.1. *Let K_π, K_ρ be connection maps for $\pi : E \to M$ and $\rho : F \to M$, then $K_\pi \oplus K_\rho$ is a connection map for $\pi \oplus \rho : E \oplus F \to M$. In a local trivialization the local connector*

$$\Gamma_{\pi\oplus\rho} : \phi U \to L(M, E \times F ; E \times F)$$

is given by

$$\Gamma_{\pi\oplus\rho}(x) \cdot (y, (\xi, \eta)) = (\Gamma_\pi(x) \cdot (y, \xi), \Gamma_\rho(x) \cdot (y, \eta)).$$

Proof. We use here the identification from Lemma 1.1 and the Proposition follows from the local formula, which is evident. q.e.d.

Given two bundles $\pi : E \to M$, $\rho : F \to M$, we have the bundle $L(\pi, \rho)$: $L(E, F) \to M$ of bounded linear maps of fibres: $L(E, F)_p = L(E_p, F_p)$. We have moreover a bundle $L(\tau^*\pi, T\rho)$: $L(\tau^*E, TF) \to TM$, with fibre $L(\tau^*E, TF)_v = L(E_{\tau(v)}, T_vF)$, $T_vF = (T\rho)^{-1}(v)$. Here $\tau^*\pi$: $\tau^*E \to TM$ is the pull back of $\tau : TM \to M$, t.i. $(\tau^*E)_v = E_{\tau(v)}$. Let (U, ϕ, Φ_π) and (U, ϕ, Φ_ρ) be local trivializations for π and ρ, we then have induced trivializations:

$$\Phi_{L(\pi,\rho)} : L(\pi, \rho)^{-1}(U) \to \phi U \times L(E, F) ,$$
$$\Phi_{L(\pi,\rho)p}(A) \cdot \xi = \Phi_{\rho p}(A \cdot \Phi_{\pi p}^{-1}(\xi)),\ A \in L(E_p, F_p),\ \xi \in E ,$$
$$\Phi_{L(\tau^*\pi, T\rho)} : L(\tau^*\pi, T\rho)^{-1}(U) \to \phi U \times M \times L(E, F \times F) ,$$
$$\Phi_{L(\tau^*\pi, T\rho)v}(A) \cdot \xi = P(T\Phi_\rho(A \cdot \Phi_{\pi p}^{-1}(\xi))) ,$$

where

$$P(x, \xi, y, \eta) = (\xi, \eta), \; A \in L(E_p, T_vF), \; p = \tau(v).$$

Then

$$\Phi_{L(\tau^*\pi, T\rho)}(A) = (T\phi(v), \Phi_{L(\tau^*\pi, T\rho)v}(A)).$$

Lemma 1.2. *Let K be a connection map for $\pi : E \to M$, then there is a canonically induced bundle equivalence :*

$$\begin{array}{ccc} TL(E, F) & \xrightarrow{L(K)} & L(\tau^*E, TF) \\ TL(\pi, \rho) \downarrow & & \downarrow L(\tau^*\pi, T\rho) \\ TM & = & TM \end{array}$$

In a local trivialization, $L(K)$ is given by

$$L(K)_\phi(x, y, A, B) = (x, y, A, B - A \cdot \Gamma(x) \cdot (y, \cdot)).$$

Proof. We will first define $L(K)$ intrinsically and then show that it has the required properties by writing it down locally. Let $W \in TL(E, F)$ and put $v = TL(\pi, \rho)(W)$, $p = \tau(v)$. Assume W, v are tangent and let c be a C^1-curve in M with $\partial c(0) = v$ and β a field along c in $L(E, F)$ with $\partial\beta(0) = W$. We have to define $L(K)(W)$ as a linear map $E_p \to T_vF$. Let $\xi \in E_p$ and α the parallel field along c with $\alpha(0) = \xi$. Put $\gamma(t) = \beta(t) \cdot \alpha(t)$, then γ is a curve in F. We define $L(K)(W) \cdot \xi = \partial\gamma(0)$. To compute this in a local trivialization, let $T\Phi_{L(\pi,\rho)}(W) = (x, A, y, B)$, then $\phi(p) = x$, $T\phi(v) = (x, y)$. Put $\Phi_\pi(\xi) = (x, \eta)$ and let $\boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{\alpha}$ be the principal parts, then $\boldsymbol{c}(0) = x$, $\boldsymbol{\beta}(0) = A$, $\boldsymbol{\alpha}(0) = \eta$, $\boldsymbol{c}'(0) = y$, $\boldsymbol{\beta}'(0) = B$. We have $\Phi_\rho \circ \gamma = (\boldsymbol{c}, \boldsymbol{\beta} \cdot \boldsymbol{\alpha})$ and

$$T\Phi_\rho \circ \partial\gamma(0) = (x, A \cdot \eta, y, B \cdot \eta - A \cdot \Gamma(x) \cdot (y, \eta))$$

as $\boldsymbol{\alpha}'(t) + \Gamma(\boldsymbol{c}(t)) \cdot (\boldsymbol{c}'(t), \boldsymbol{\alpha}(t)) = 0$, where Γ is the local connector for K. We have then proved the local formula which shows that $L(K)$ is well defined and gives a bundle equivalence.

Proposition 1.2. *Let K_π, K_ρ be connection maps for the vector bundles $\pi : E \to M$ and $\rho : F \to M$. Then $K_{L(\pi,\rho)} = L(\pi, K_\rho) \circ L(K_\pi)$ is a connection map for $L(\pi, \rho) : L(E, F) \to M$. In a local trivialization the corresponding local connector is given by*

$$\Gamma_{L(\pi,\rho)} : \rho U \to L(M, L(E, F) \,;\, L(E, F)),$$
$$[\Gamma_{L(\pi,\rho)}(x) \cdot (y, A)] \cdot \xi = \Gamma_\rho(x) \cdot (y, A \cdot \xi) - A \cdot \Gamma_\pi(x) \cdot (y, \xi).$$

Remark. Here $L(\pi, K_\rho) : L(\tau^*E, TF) \to L(E, F)$ is defined by

$$L(\pi, K_\rho)(A) \cdot \xi = K_\rho(A \cdot \xi).$$

Proof. Using Lemma 1.2, we have in a local trivialization :

$$K_{L(\pi,\rho)\phi}(x, A, y, B) \cdot \xi = K_{\rho\phi}(x, A \cdot \xi, y, B \cdot \xi - A \cdot \Gamma_\pi(x) \cdot (y, \xi))$$
$$= (x, B \cdot \xi - A \cdot \Gamma_\pi(x) \cdot (y, \xi) + \Gamma_\rho(x) \cdot (y, A \cdot \xi)) .$$

Obviously $\Gamma_{L(\pi,\rho)}$ is of class C^{r-1} if Γ_ρ and Γ_π are.

2. Covariant differentiation in vector bundles

We will assume that we have a connection on M, t.i. a connection map K_τ for the tangent bundle $\tau : TM \to M$. Furthermore, let $\pi : E \to M$ be a vector bundle with connection map K_π. If ξ is a differentiable section in π, we will define the covariant derivative of ξ to be the section in $L(\tau, \pi) : L(TM, E) \to M$ defined by

$$\nabla\xi(p) = K_\pi \circ T_p\xi, \ \mathrm{T}_p\xi = T\xi \,|\, T_pM .$$

In a local trivialization we have

$$\nabla\xi(x) \cdot y = D\xi(x) \cdot y + \Gamma_\pi(x) \cdot (y, \xi(x)) ,$$

where we have used the same letter for the principal part and Γ_π is the local connector. If ξ is of class C^s, then $\nabla\xi$ is obviously of class C^{s-1} for $1 \leq s \leq r-1$, where r is the class of E as before. We then define higher order covariant derivatives inductively, using the induced connection on $L^t(TM, E) = \mathrm{L}(TM, L^{t-1}(TM, E))$ by Proposition 1.2. Here the connection on M is needed. So $\nabla^t\xi$ is a section in $L^t(\tau, \pi)$. Let $\rho : F \to M$ be another vector bundle of the same class as π and A, ξ sections in $L(\pi, \rho)$, π. We then define a section $A \cdot \xi$ in ρ by $(A \cdot \xi)(p) = A(p) \cdot \xi(p)$. If ξ, X are sections in π and τ, we will call $\nabla\xi \cdot X$ the partial derivative of ξ in the direction X; it is again a section in π. In the case $\pi = \tau$, we have the classical covariant differentiation of vector fields.

Lemma 2.1. (i) *Let π and ρ be vector bundles over M with connection and give $L(\pi, \rho)$ the induced connection. Then for any sections A, ξ in $L(\pi, \rho)$ and π of class C^1, we have*

$$\nabla(A \cdot \xi) = \nabla A \cdot (\cdot\,, \xi) + A \cdot \nabla\xi .$$

(ii) *If π is a bundle of class C^r with C^{r-1} connection there is a C^{r-2} section R in $L(\tau, \tau, \pi; \pi)$, such that for any sections X, Y in τ and ξ of class C^2 in π, we have*

$$\nabla^2\xi \cdot (X, Y) - \nabla^2\xi \cdot (Y, X) = R \cdot (X, Y, \xi) .$$

In any local trivialization we have

$$R(x) \cdot (y, z, \xi) = D\Gamma(x) \cdot (y, z, \xi) - D\Gamma(x) \cdot (z, y, \xi) + \Gamma(x) \cdot (y, \Gamma(x) \cdot (z, \xi)) - \Gamma(x) \cdot (z, \Gamma(x) \cdot (y, \xi)).$$

Remark. In fact this is a generalization of the classical curvature tensor, so we may keep this name for R. In the formula above $\Gamma = \Gamma_\pi$ is the local connector for π.

Proof. We have only to write this down in a local trivialization and use the formula in Proposition 1.2 for the local connector of $L(\pi, \rho)$ and $L(\tau, \pi)$.

Using the same formula we get easily

Lemma 2.2. *Let R_π, R_ρ be the curvature tensors for bundles π and ρ over M with some connections, and take the induced connection for $L(\pi, \rho)$. Then the curvature tensor for $L(\pi, \rho)$ is given by the formula*

$$R_{L(\pi,\rho)}(p) \cdot (v, u, A) \cdot \xi = R_\rho(p) \cdot (v, u, A \cdot \xi) - A \cdot R_\pi(p) \cdot (v, u, \xi),$$

for $v, u \in T_pM$, $\xi \in E_p$, $A \in L(E_p, F_p)$.

Let $\pi: E \to M$, $\rho: F \to M$ be vector bundles with connection maps K_π, K_ρ and let $f: E \to F$ be a fibre map, t.i. $\rho \circ f = \pi$. We then define the covariant derivative of f as a fibre map:

$$\nabla f: E \to L(TM \oplus E, F)$$

by

$$\nabla f(\xi) \cdot (v, \eta) = K_\rho \circ Tf \circ (\pi_1, T\pi, K_\pi)^{-1}(\xi, v, \eta).$$

In a local trivialization, we get easily the expression

$$D_1 f(x, \xi) \cdot y + D_2 f(x, \xi) \cdot (\eta - \Gamma_\pi(x) \cdot (y, \xi)) + \Gamma_\rho(x) \cdot (y, f(x, \xi))$$

for the principal part, which shows that $\nabla f(\xi)$ is in fact a linear map: $T_pM \oplus E_p \to F_p$ for $\xi \in E_p$. Moreover it reveals, that if we split ∇f in the obvious manner into $\nabla_1 f: E \to L(TM, F)$ and $\nabla_2 f: E \to L(E, F)$, then $\nabla_2 f(\xi) = Df_p(\xi)$, where $f_p = f | E_p$ (note that $f_p: E_p \to F_p$ is a map from a Banach space into a Banach space). We will therefore feel free to write $D_2 f$ instead of $\nabla_2 f$. Using the local formula above the following is easily seen:

Lemma 2.3. *Let $f: E \to F$ be a differentiable fibre map and ξ a section in E, then we have*

$$\nabla(f \circ \xi)(p) = \nabla_1 f(\xi(p)) + D_2 f(\xi(p)) \circ \nabla \xi(p).$$

These formulas show us that covariant differentiation behaves very much alike ordinary differentiation in Banach spaces. We will now see how it behaves under pull-backs.

Let M, M' be Banach manifolds of class C^k as before and let $\pi: E \to M'$ be a vector bundle of class C^r, $1 \leq r \leq k$, with a connection map $K': TE \to E$.

Let $h: M \to M'$ be a map of class C^r. Then the pull-back $h^*\pi: h^*E \to M$, $(h^*E)_p = E_{h(p)}$, is a bundle of class C^r. If (U, ϕ) is a chart for M and (V, ψ, Ψ) a local trivialization for π, such that $h(U) \subset V$, then we can define a local trivialization for $h^*\pi$ by

$$\Phi : h^*E | U \to \phi U \times E\,;\ \Phi(\xi_p) = (\phi(p), \Psi_{h(p)}(\xi_p))\,,$$

where E is the fibre model for π in the given trivialization.

We now define a connection map K for $h^*\pi$ by $i \circ K = K' \circ Ti$, where i: $h^*E \to E$ is the inclusion. This defines K uniquely as i is an isomorphism on each fibre.

For the local connector we get

$$\Gamma(x) \cdot (y, \xi) = \Gamma'(h_0(x)) \cdot (Dh_0(x) \cdot y, \xi)\,,$$

where Γ' is the local connetor for π and h_0 the local representative for h. We can regard the tangent of h as a section ∂h in $L(\tau, h^*\tau')$ by $\partial h(p) \cdot v = Th(v)$. The sections in $h^*\pi$ are in a class preserving one-to-one correspondence with the fields along h, t.i. maps $\xi : M \to E$ s.t. $\pi \circ \xi = h$ and we will identify those. A section X in π induces a section $h^*X = X \circ h$ in $h^*\pi$ and it follows easily from the above local formula for the connectors that

$$(h^*\nabla')(h^*X) = h^*(\nabla' X) \cdot \partial h\,,$$

where $h^*\nabla'$, ∇' denote the covariant differentiation in $h^*\pi$, π. Now let M be with connection and let $\nabla = h^*\nabla'$ denote the covariant differentiation over M. Then $\nabla \partial h$ is symmetric bilinear on each fibre, so by using Lemma 2.1 (i) and (ii) for the curvature tensor R_h of $h^*\pi$ we get

$$R_h \cdot (v, u, \xi) = (R' \circ h) \cdot (\partial h \cdot v, \partial h \cdot u, \xi)\,,$$

where R' is the curvature tensor for π.

Remark. We have constantly assumed the connection for a manifold to be symmetric (without torsion) t.i. the local connector is a symmetric bilinear map at each point. This is no restriction on existence.

3. Connection in iterated tangent bundles

Here M is to be a Banach manifold of class C^k, $k \geq 2$, without a boundary and with a connection map $K : T^2M \to TM$ for the tangent bundle τ. We will assume that the connection is symmetric. We define a map $K_T : T^3M \to T^2M$ through the properties :

$$K_T1 : \tau_1 \circ K_T = \tau_1 \circ \tau_2\,, \quad K_T2 : T\tau \circ K_T = K \circ T^2\tau\,,$$
$$K_T3 : K \circ K_T = K \circ TK - R \circ (T\tau \circ T^2\tau,\ \tau_1 \circ T\tau_1,\ \tau_1 \circ T^2\tau)\,.$$

Here $\tau_i : T^{i+1}M \to T^iM$ are the iterated tangent bundles and R is the curvature tensor, defined as a section in $L^3(TM, TM)$ in Lemma 2.1, but considered as a trilinear bundle map $R : TM \oplus TM \oplus TM \to TM$ here. We have observed that $(\tau_1, T\tau, K)$ is a diffeomorphism, so K_T is uniquely defined by those properties.

Theorem 3.1. *The map K_T is a connection map for the vector bundle τ_1: $T^2M \to TM$ and gives the manifold TM a symmetric connection of class C^{k-3}. In any local trivialization induced by a chart (U, ϕ) for M, the local connector for K_T, Γ_T: $\phi U \times M \to L(M \times M, M \times M; M \times M)$ is given by the formula*

$$\Gamma_T(x, y) \cdot ((z_1, z_2), (\xi_1, \xi_2)) = (\Gamma(x) \cdot (z_1, \xi_1), D\Gamma(x) \cdot (y, z_1, \xi_1) + \Gamma(x) \cdot (z_2, \xi_1) + \Gamma(x) \cdot (z_1, \xi_2)),$$

where Γ is the local connector for K.

Proof. We have only to prove the local formula which is a simple computation, using the formula for the curvature tensor from Lemma 2.1. Γ_T is obviously of class C^{k-3} as Γ is of class C^{k-2}.

We will denote by ∇_T the covariant differentiation in vector bundles over TM, using the connection map K_T for the tangent bundle of TM and by ∇ as before the covariant differentiation over M. A field α along a curve c in M is called a Jacobi field, iff

$$\nabla^2\alpha + (R \circ c) \cdot (\alpha, \partial c, \partial c) = 0 .$$

Theorem 3.2. *A curve α of class C^2 in TM is the geodesic ($\nabla_T\partial\alpha = 0$) in TM with $\partial\alpha(0) = \omega$, iff α is the Jacobi field along the geodesic $c = \tau \circ \alpha$ in M with $\partial c(0) = T\tau(\omega)$, $\alpha(0) = \tau_1(\omega)$ and $\nabla\alpha(0) = K(\omega)$.*

Proof. 1) Suppose α is a geodesic in TM: $\nabla_T\partial\alpha = 0$, $\partial\alpha(0) = \omega$. Then we have for $c = \tau \circ \alpha$, $\partial c = Tc \circ e = T\tau \circ \partial\alpha$, so $\partial c(0) = T\tau(\omega)$ and

$$\nabla\partial c = K \circ T\partial c \circ e = K \circ T^2\tau \circ T\partial\alpha \circ e = T\tau \circ K_T \circ \partial^2\alpha = T\tau \circ \nabla_T\partial\alpha = 0 .$$

Furthermore, $\nabla\alpha = K \circ \partial\alpha$ and then

$$\nabla^2\alpha = K \circ T(K \circ \partial\alpha) \circ e = (K \circ K_T + R \circ (T\tau \circ T^2\tau, \tau_1 \circ T\tau_1, \tau_1 \circ T^2\tau)) \circ \partial^2\alpha = R \circ (\partial c, \alpha, \partial c) = - R \circ (\alpha, \partial c, \partial c),$$

which, together with $\nabla_T\partial\alpha = 0$ and $T\tau \circ \partial\alpha = T(\tau \circ \alpha) \circ e = \partial c$, $\tau_1 \circ \alpha = c$, $\tau \circ \partial c = c$, proves that α is a Jacobi field along c. We have $\alpha(0) = \tau_1 \circ \partial\alpha(0) = \tau_1(\omega)$ and $\nabla\alpha(0) = K \circ \partial\alpha(0) = K(\omega)$.

2) Now suppose α is a Jacobi field along the geodesic $c = \tau \circ \alpha$ and put $\omega = (\tau_1, T\tau_1 K)^{-1}(\alpha(0), \partial c(0), \nabla\alpha(0))$. Then we have

$$\begin{aligned} K \circ \nabla_T \partial\alpha &= K \circ K_T \circ \partial^2\alpha \\ &= (K \circ TK - R \circ (T\tau \circ T^2\tau,\ \tau_1 \circ T\tau_1,\ \tau_1 \circ T^2\tau)) \circ \partial^2\alpha \\ &= \nabla^2\alpha - R \circ (\partial c, \alpha\, \partial c) = 0\,, \\ T\tau \circ \nabla_T \partial\alpha &= T\tau \circ K_T \circ \partial^2\alpha = K \circ T^2\tau \circ T\partial\alpha \circ e \\ &= K \circ T\partial c \circ e = \nabla\partial c = 0\,. \end{aligned}$$

Now as $\ker K \cap \ker T\tau = 0$, we get $\nabla_T \partial\alpha = 0$. Moreover from $K \circ \partial\alpha(0) = \nabla\alpha(0) = K(\omega)$, $T\tau \circ \partial\alpha(0) = \partial c(0) = T\tau(\omega)$ we get $\partial\alpha(0) = \omega$. q.e.d.

We have now to investigate the exponential maps exp, $\exp_T$ corresponding to ∇ and ∇_T for later use. Moreover we have to analyze the two first derivatives of exp. The general reference on local existence, uniqueness and differentiability of flows in Banach manifolds is Lang [6].

There is an open neighborhood $\mathcal{O}$ of the set of zero vectors in TM and a map $\exp : \mathcal{O} \to M$ of class C^{k-2}, if the class k of M is ≥ 3. We have $\exp v = c(1)$, where c is the unique geodesic c: $[0, 1] \to M$ with $\partial c(0) = v$. $\mathcal{O}$ is the set of $v \in TM$ such that c is defined on the unite inverval, then $c(t) = \exp tv$, $0 \leq t \leq 1$. We define the covariant derivative of exp :

$$\nabla \exp : \mathcal{O} \oplus TM \oplus TM \to TM$$

by

$$\nabla \exp = T \exp \circ (\tau_1, T\tau, K)^{-1}\,,$$

which is then a map of class C^{k-3}.

Theorem 3.3. *Let $v \in \mathcal{O}_p$ and $u, w \in T_pM$. Let $c(t) = \exp tv$ and Y the Jacobi field along c with $Y(0) = u$, $\nabla Y(0) = w$. Then*

$$Y(t) = \nabla \exp (tv, u, tw)\,, \quad 0 \leq t \leq 1\,.$$

Proof. Let b: $[0, \epsilon] \to M$ be a curve of class C^k in M with $\partial b(0) = u$ and β a field of class C^{k-1} along b with $\partial\beta(0) = v$ and $\nabla\beta(0) = w$ (e.g. geodesic and Jacobi field). Put $\alpha(t, s) = \exp(t\beta(s))$. Then we have

$$\begin{aligned} \partial_2\alpha(t, s) &= \nabla \exp (t\beta(s), \partial b(s), t\nabla\beta(s))\,, \\ \nabla_1\partial_2\alpha &= \nabla_2\partial_1\alpha\,, \\ \nabla_1^2\partial_2\alpha &= \nabla_2\nabla_1\partial_1\alpha + (R \circ \alpha) \cdot (\partial_1\alpha, \partial_2\alpha, \partial_1\alpha)\,. \end{aligned}$$

The last formula follows from Lemmas 2.1 and 2.2, if we think of $\partial\alpha$ as a section in the pull-back $\alpha^*\tau$, introduce partial derivatives in the obvious manner and make use of the formula for the pull-back of the curvature tensor. Now as $\nabla_1\partial_1\alpha \equiv 0$, it follows that $t \to \alpha(t, s)$ is a Jacobi field for all s, and the theorem then follows for $s = 0$ as $\alpha(t, 0) = c(t)$, $\partial_2\alpha(0, 0) = \partial b(0) = u$,

$\nabla_1\partial_2\alpha(0, 0) = \nabla\beta(0) = w$. The theorem and this variation of geodesic through geodesics with Jacobi fields as variation field is well known in differential geometry, at least for $u = 0$.

Define $\sigma = (\tau_1, T\tau, K)^{-1} \circ (T\tau, \tau_1, K) : T^2M \to T^2M$, locally we have $\sigma_\phi(x, y, \xi, \eta) = (x, \xi, y, \eta)$. Combining the two last theorems we obtain

Corollary 3.1. $\exp_T = T\exp \circ \sigma$.

We define the second covariant derivative of exp

$$\nabla^2 \exp : \mathcal{O} \oplus (\oplus^\sigma TM) \to TM$$

by

$$\begin{aligned}\nabla^2 \exp\,(v, u, w, \xi_0, \xi_1, \xi_2, \xi_3) \\ = K(T\nabla\exp(A(v, \xi_0, \xi_1), A(u, \xi_0, \xi_2), A(w, \xi_0, \xi_3)))\,,\end{aligned}$$

with $A = (\tau_1, T\tau, K)^{-1}$. We have here used the identification in Lemma 1.1 of the tangent of a direct sum with the direct sum of the tangents.

Theorem 3.4. *Let $v \in \mathcal{O}_p$, $u, w, \xi_0, \xi_1, \xi_2, \xi_3 \in T_pM$. Put $c(t) = \exp tv$, $Y_1(t) = \nabla \exp\,(tv, u, tw)$, $Y_2(t) = \nabla \exp\,(tv, \xi_0, t\xi_1)$. Let $Z(t)$ be the solution of the initial value problem :*

$$\begin{aligned}\nabla^2 Z + (R \circ c)\cdot(Z, \partial c, \partial c) &= \frac{1}{2}(\nabla R \circ c)\cdot(\partial c, \partial c, Y_1, Y_2) \\ &+ \frac{1}{2}(\nabla R \circ c)\cdot(\partial c, \partial c, Y_2, Y_1) + 2(R \circ c)\cdot(\partial c, Y_2, \nabla Y_1) \\ &+ 2(R \circ c)\cdot(\partial c, Y_1, \nabla Y_2)\,,\end{aligned}$$

with $Z(0) = \xi_2$, $\nabla Z(0) = \xi_3 + R(p)\cdot(v, \xi_0, u)$. Then

$$Z(t) = \nabla^2 \exp\,(tv, u, tw, \xi_0, t\xi_1, \xi_2, t\xi_3)\,.$$

Proof. Let $b : [0, \varepsilon] \to M$ be a C^k-curve with $\partial b(0) = \xi_0$ and $\beta_1, \beta_2, \beta_3$ C^{k-1}-fields along b with $\beta_1(0) = v$, $\beta_2(0) = u$, $\beta_3(0) = w$ and $\nabla\beta_i(0) = \xi_i$, $i = 1, 2, 3$. We put

$$\alpha(t, s) = \nabla \exp\,(t\beta_1(s), \beta_2(s), t\beta_3(s))\,.$$

Then

$$\nabla_2\alpha(t, s) = \nabla^2 \exp\,(t\beta_1(s), \beta_2(s), t\beta_3(s), \partial b(s), t\nabla\beta_1(s), \nabla\beta_2(s), t\nabla\beta_3(s))\,.$$

With $a(t, s) = \tau \circ \alpha(t, s) = \exp\,(t\beta_1(s))$, we have by Theorem 3.3, $\nabla_1^2\alpha = -(R \circ a)\cdot(\alpha, \partial_1 a, \partial_1 a)$ and then

$$\nabla_1\nabla_2\alpha = \nabla_2\nabla_1\alpha + (R \circ a)\cdot(\partial_1 a, \partial_2 a, \alpha)\,,$$

$$\begin{aligned}\nabla_1^2\nabla_2\alpha &= \nabla_2\nabla_1^2\alpha + (R\circ a)\cdot(\partial_1 a, \partial_2 a, \nabla_1\alpha) + \nabla_1((R\circ a)\cdot(\partial_1 a, \partial_2 a, \alpha)) \\ &= -(\nabla R\circ a)(\partial_2 a, \alpha, \partial_1 a, \partial_1 a) + (\nabla R\circ a)\cdot(\partial_1 a, \partial_1 a, \partial_2 a, \alpha) \\ &\quad + 2(R\circ a)\cdot(\partial_1 a, \partial_2 a, \nabla_1\alpha) + 2(R\circ a)\cdot(\partial_1 a, \alpha, \nabla_1\partial_2 a) \\ &\quad - (R\circ a)\cdot(\nabla_2\alpha, \partial_1 a, \partial_1 a)\,.\end{aligned}$$

We have used Lemma 2.1 for the covariant derivation and Bianchi's first identity for the curvature tensor (the cyclic sum is zero). Using Bianchi's second identity for the covariant derivative ∇R (the cyclic sum in three first variables with the fourth fixed is zero) together with the first, we get

$$\begin{aligned}\nabla R\cdot(x, y, v, v) &= \nabla R\cdot(v, y, x, v) + \nabla R\cdot(x, v, y, v) \\ &= \nabla R\cdot(v, v, x, y) + \nabla R\cdot(v, y, v, x) \\ &\quad - \nabla R\cdot(x, y, v, v)\,,\end{aligned}$$

so $2\nabla R\cdot(x, y, v, v) = \nabla R\cdot(v, v, x, y) - \nabla R\cdot(v, v, y, x)$. It now follows that $Z_s(t) = \nabla_2\alpha(t, s)$ satisfies the differential equation for Z for all s, with c, Y_1, Y_2 replaced by $a, \alpha, \partial_2 a$. But α and $\partial_2 a$ are Jacobi fields in t, for fixed s, and take the initial values of Y_1 and Y_2 for $s = 0$; moreover $Z_0(0) = \nabla\beta_2(0) = \xi_2$ and $\nabla Z_0(0) = \nabla_1\nabla_2\alpha(0, 0) = \nabla\beta_3(0) + R(p)\cdot(\partial c(0), Y_2(0), Y_1(0)) = \xi_3 + R(p)\cdot(v, \xi_0, u)$. Then by uniqueness $Z_0 = Z$. q.e.d.

It follows from the differential equation for Jacobi fields that $\nabla\exp(v, u, w)$ is linear in (u, w), so we have a splitting:

$$\nabla\exp(v, u, w) = \nabla_1\exp(v, u) + \nabla_2\exp(v, w)\,.$$

Similarly we see that $\nabla^2\exp(v, u, w, \xi_0, \xi_1, \xi_2, \xi_3)$ is linear in (u, w, ξ_2, ξ_3), so after the obvious definitions, we have

$$\begin{aligned}\nabla^2\exp(v, u, w, \xi_0, \xi_1, \xi_2, \xi_3) &= \nabla\nabla_1\exp(v, u, \xi_0, \xi_1, \xi_2) \\ &\quad + \nabla\nabla_2\exp(v, w, \xi_0, \xi_1, \xi_3)\,.\end{aligned}$$

Moreover putting $w = \xi_3 = 0$ and $u = \xi_2 = 0$ respectively in the differential equation for Z in Theorem 3.4 we see that the first and second terms on the right hand side above are linear in (ξ_0, ξ_1, ξ_2) and (ξ_0, ξ_1, ξ_3) respectively. Therefore

$$\begin{aligned}\nabla\nabla_1\exp(v, u, \xi_1, \xi_2, \xi_3) &= \textstyle\sum_{i=1}^3 \nabla_i\nabla_1\exp(v, u, \xi_i)\,, \\ \nabla\nabla_2\exp(v, w, \eta_1, \eta_2, \eta_3) &= \textstyle\sum_{i=1}^3 \nabla_i\nabla_2\exp(v, w, \eta_i)\,.\end{aligned}$$

Corollary 3.2.

a) $\nabla_i\exp(0, \xi) = \xi,\ i = 1,2\,.$

b) $\nabla_3\nabla_i\exp(v, \xi, \eta) = \nabla_i\exp(v, \eta),\ i = 1,2\,.$

c) $\nabla_1\nabla_2\exp(v, w, \xi) = \nabla_2\nabla_1\exp(v, \xi, w)$, *bilinear in* (w, ξ).

d) $\nabla_2\nabla_2 \exp(v, \xi, \eta) = \nabla_2\nabla_2 \exp(v, \eta, \xi)$, *bilinear in* (ξ, η).

e) $\nabla_j\nabla_i \exp(0, \xi, \eta) = 0$, $i, j = 1,2$.

This follows at once from the differential equations. Note that the equation in Theorem 3.4 is symmetric in Y_1, Y_2.

The restriction of $\nabla_2 \exp$ to $T_xM \oplus T_xM$ for some $x \in M$ is just the tangent of $\exp_x = \exp | T_xM$. Using a) above and the inverse function Theorem there is an open neighborhood $D_x \subset T_xM$ of ox the zero vector at x, such that $\exp_x$ maps D_x diffeomorphic into M. It follows that there is an open neighborhood $\mathscr{D} \subset TM$ of the set of zero vectors in TM, such that $(\tau, \exp)$ maps $\mathscr{D}$ diffeomorphic onto an open neighborhood of the diagonal in $M \times M$. Then $\nabla_2 \exp(v, \cdot) : T_xM \to T_{\exp v}M$ is a toplinear isomorphism for every $v \in \mathscr{D}$, $x = \tau(v)$. We therefore have a map $\vartheta : \mathscr{D} \to L(TM, TM)$ by

$$\nabla_2 \exp(v, \vartheta(v) \cdot u) = \nabla_1 \exp(v, u),$$

which is a fibre preserving map of class C^{k-3}. Similarly we define a map $\Lambda : \mathscr{D} \to L^2(TM, TM)$ by

$$\nabla_2\nabla_2 \exp(v, w, \xi) = \nabla_2 \exp(v, \Lambda(v) \cdot (w, \xi)).$$

Then Λ is a fibre preserving map of class C^{k-4} and maps actually into the subbundle of bilinear symmetric maps by d) Corollary 3.1.

Lemma 3.1.

a) $\vartheta(0) = id$, $D_2\vartheta(0) = 0$, $\vartheta(v) \cdot v = v$.

b) $\Lambda(0) = 0$, $\Lambda(v) \cdot (v, v) = 0$, $v \in \mathscr{D}$.

c) $\nabla_1\nabla_2 \exp(v, w, \xi) = \nabla_2 \exp(v, D_2\vartheta(v) \cdot (w, \xi) + \Lambda(v) \cdot (\vartheta(v) \cdot \xi, w))$.

Proof. $\vartheta(0)$ is the identity by a) Corollary 3.1. If we replace v by $v + tw$ and take the derivative of both sides with respect to t, we obtain for $t = 0$:

$$\nabla_2\nabla_1 \exp(v, u, w) = \nabla_2\nabla_2 \exp(v, \vartheta(v) \cdot u, w) + \nabla_2 \exp(v, D_2\vartheta(v) \cdot (w, u)).$$

Then using Corollary 3.1 c) we get c) and putting $v = 0$ there and in the defining equation for Λ gives $D_2\vartheta(0) = \Lambda(0) = 0$. We have $\nabla_2 \exp(tv, tv) = t\partial c$, where $c = \exp tv$. Therefore, if we put $w = \xi_1 = v$, $u = \xi_0 = \xi_2 = \xi_3 = 0$ in the differential equation in Theorem 3.4 we have $Y_1 = Y_2 = t\partial c$, $\nabla Y_1 = \nabla Y_2 = \partial c$, so the equation reduces to the Jacobi equation with zero initial conditions, thus the solution is zero which means $\nabla_2\nabla_2 \exp(v, v, v) = 0$, so we have $\Lambda(v) \cdot (v, v) = 0$. q.e.d.

The result of our effort is that a complete knowledge of the two first derivatives of the exponential map is now stored in $\nabla_2 \exp$, $\nabla_2\nabla_2 \exp$ and the fibre maps ϑ and Λ, both of them will play an important role in the theory of manifolds of maps.

4. Banach spaces of sections

Here N will denote a compact Hausdorff space, whose topology has a countable base. Then N is normal and metrizable. We call N of class C^r, $r \geq 1$, iff N is a C^r-differentiable manifold with (or without) a boundary, else we call N of class C^0 (for convenience only).

Let $\pi : E \to N$ be a vector bundle of class C^s, $0 \leq s \leq r$, where r is the class of N. We denote by $C^s(\pi)$ the linear space of sections of class C^s in π and we will define a normable topology in $C^s(\pi)$ in the following way:

We take any Finsler structure for π, t.i. a continuous function $\|\cdot\| : E \to R$, which is an admissible norm on each fibre. Clearly Finsler structures exist and any two are equivalent as N is compact. We then define a norm for $C^0(\pi)$ by

$$\|\xi\|_{C^0} = \sup \|\xi(p)\| \, (p \in N) .$$

For $r \geq s \geq 1$, we take a Finsler structure for the tangent bundle $\tau : TN \to N$ and connection of class C^{s-1} for π. Moreover if $r \geq 2$ we take a connection of class C^{r-2} for τ. We then have an induced Finsler structure for

$$L^j(\tau, \pi), j \geq 1 : \|A\| = \sup \|A \cdot (v_1, \cdots, v_j)\| \quad (\|v_i\| = 1, 1 \leq i \leq j) .$$

This gives us a norm for $C^0(L^j(\tau, \pi))$ as before. For $r \geq 2$ we take the induced connection for $L^j(\tau, \pi)$ as in Proposition 1.2 and define

$$\|\xi\|_{C^s} = \textstyle\sum_{j=0}^{s} \|\nabla^j \xi\|_{C^0} ,$$

where ∇ denotes covariant differentiation. Note that we need a connection for $L(\tau, \pi)$ only for $s \geq 2$. It is well known that $C^s(\pi)$ with this norm is a Banach space, and it can easily be shown that we get equivalent norms, if we change the structures used for the construction (N being compact!).

Let $\mathfrak{B}$ be the category of real Banach spaces and $VB(N)$ the category of vector bundles of class C^r over N with fibres in $\mathfrak{B}$. The set of morphisms $\pi \to \rho$ is then the Banachable space $C^r(L(\pi, \rho))$. Then C^s is a covariant functor $C^s : VB(N) \to \mathfrak{B}$, with

$$C^s_* : C^r(L(\pi, \rho)) \to L(C^s(\pi), C^s(\rho))$$

by $C^s_*(A) \cdot \xi = A \cdot \xi$. C^s_* is a continuous linear map for any $s \leq r$ as is easily seen using Lemma 2.1(i). We will consider C^s_* as inclusion map because of naturality. Let $\mathfrak{A}$ be a full subcategory of $\mathfrak{B}$ closed under the operations of taking direct sums (products) and bounded linear maps. (e.g. $\mathfrak{A} = \mathfrak{B}$ or finite dimensional spaces.) Let $VB(N, \mathfrak{A})$ denote the category of vector bundles of class C^r over N with fibres in $\mathfrak{A}$. By a section functor $\mathfrak{T}$ on $VB(N, \mathfrak{A})$, we mean a covariant functor

$$\mathfrak{T} : VB(N, \mathfrak{A}) \to \mathfrak{B}$$

which assignes to every $\pi \in VB(N, \mathfrak{A})$ a Banachable space $\mathfrak{T}(\pi)$ of sections in π and where the induced map of morphisms:

$$\mathfrak{T}_* : C^r(L(\pi, \rho)) \to L(\mathfrak{T}(\pi), \mathfrak{T}(\rho))$$

is a continuous linear inclusion, $\mathfrak{T}_*(A) \cdot \xi = A \cdot \xi$. Given three section functors $\mathfrak{T}_i$, we will say, that the relation $\mathfrak{T}_1 L \subset L(\mathfrak{T}_2, \mathfrak{T}_3)$ holds, if for any two bundles π, ρ from $VB(N, \mathfrak{A})$ we have a continuous linear inclusion

$$\mathfrak{T}_1(L(\pi, \rho)) \subset L(\mathfrak{T}_2(\pi), \mathfrak{T}_3(\rho))$$

defined as above by $(A \cdot \xi)(p) = A(p) \cdot \xi(p)$. This means $A \cdot \xi \in \mathfrak{T}_3(\rho)$ for $A \in \mathfrak{T}_1(L(\pi, \rho))$, $\xi \in \mathfrak{T}_2(\pi)$ and taking any admissible norms, there is a constant $C > 0$ s.t.

$$||A \cdot \xi||_{\mathfrak{T}_3} \leq C\, ||A||_{\mathfrak{T}_1} ||\xi||_{\mathfrak{T}_3} .$$

In particular $C^r L \subset L(\mathfrak{T}, \mathfrak{T})$ is supposed to hold for any section functor $\mathfrak{T}$ and it then follows easily that the spaces $\mathfrak{T}(\pi \oplus \rho)$ and $\mathfrak{T}(\pi) \times \mathfrak{T}(\rho)$ are top-linearly isomorphic under the natural bijection.

We will call a section functor $\mathfrak{S}$ satisfying the following three conditions a manifold model:

1) We have a continuous linear inclusion $\mathfrak{S}(\pi) \subset C^0(\pi)$ for every $\pi \in VB(N, \mathfrak{A})$.

2) $\mathfrak{S}L \subset L(\mathfrak{S}, \mathfrak{S})$ holds.

3) Let E, $F \in VB(N, \mathfrak{A})$, $\mathcal{O} \subset E$ an open subset projected onto N and $f : \mathcal{O} \to F$ a fibre preserving map of class C^r, then for every $\xi \in \mathfrak{S}(\mathcal{O}) = \{\xi \in \mathfrak{S}(E) : \xi(N) \subset \mathcal{O}\}$ we have $f \circ \xi \in \mathfrak{S}(F)$ and the map

$$\mathfrak{S}(f) : \mathfrak{S}(\mathcal{O}) \to \mathfrak{S}(F)$$

thus defined is continuous.

Remarks. It follows from 1) and the fact that N is compact, that $\mathfrak{S}(\mathcal{O})$ is open in $\mathfrak{S}(E)$. The compactness of N is needed in order that $C^0(\mathcal{O})$ is open. It follows from 2) that $\mathfrak{S}(E)$ is an $\mathfrak{S}(L(E,E))$ module and $\mathfrak{S}(N,R) = \mathfrak{S}(N \times R)$ is a Banach algebra of functions on N. Let $f : \mathcal{O} \to F$ be a map as above in 3). We call f of class (C^r, C^{r+s}), $0 \leq s \leq \infty$, iff $f_p = f \mid \mathcal{O}_p$ is of class C^s and $D_2^i f : \mathcal{O} \to L^i(E, F)$ defined by $D_2^i F \mid \mathcal{O}_p = D^i f_p$, is of class C^r for $0 \leq i \leq s$. In particular f is of class (C^r, C^r), iff f is of class C^r and any morphism is of class $(C^r, C^{r+\infty})$.

Lemma 4.1. *Let the section functor* $\mathfrak{S}$ *be a manifold model and* $f : \mathcal{O} \to F$ *a fibre preserving map of class* (C^r, C^{r+s}), *then* $\mathfrak{S}(f)$ *is of class* C^s *and*

$$D^s\mathfrak{S}(f) = \mathfrak{S}(D_2^s f) .$$

Proof. By induction on $s \geq 0$:

The Lemma is true by assumption for $s = 0$ and it is obviously sufficient to prove it for $s = 1$, as the step from s to $s + 1$ then follows replacing f by $D_2^s f$ and F by $L^s(E, F)$. Now $D_2 f : \mathcal{O} \to L(E, F)$ is of class C^r, therefore

$$\mathfrak{S}(D_2 F) : \mathfrak{S}(\mathcal{O}) \to \mathfrak{S}(L(E, F)) \subset L(\mathfrak{S}(E), \mathfrak{S}(F))$$

is continuous by 3) and 2) above. Let $\xi \in \mathfrak{S}(\mathcal{O})$ and $\mathcal{O}' \subset \mathcal{O}$ an open neighborhood of $\xi(N)$ such that each fibre $\mathcal{O}'_p$ is convex. Such a neighborhood can easily be constructed using any Finster structure, as ξ is continuous. We define $\theta : \mathcal{O}' \oplus \mathcal{O}' \to L(E, F)$ as the fibre preserving map:

$$\theta(x, y) = \int_0^1 D_2 f(x + t(y - x))dt - D_2 f(x) ,$$

then θ is of class C^r, $\theta(x, x) = 0$ for all $x \in \mathcal{O}'$ and

$$f(y) - f(x) - D_2 f(x) \cdot (y - x) = \theta(x, y) \cdot (y - x) .$$

Therefore for any $\eta \in \mathfrak{S}(\mathcal{O}')$:

$$\mathfrak{S}(f)(\eta) - \mathfrak{S}(f)(\xi) - \mathfrak{S}(D_2 f)(\xi) \cdot (\eta - \xi) = \mathfrak{S}(\theta)(\xi, \eta) \cdot (\eta - \xi) .$$

Now $\mathfrak{S}(\theta)(\xi, \xi) = 0$ and $\mathfrak{S}(\theta)$ is continuous by property 3) of $\mathfrak{S}$, it follows that $\mathfrak{S}(f)$ is differentiable at ξ with

$$D\mathfrak{S}(f)(\xi) = \mathfrak{S}(D_2 f)(\xi) = D_2 f \circ \xi .$$

5. Manifolds of maps

Let N and $\mathfrak{A}$ be as in §4 and $\mathfrak{S}$ a manifold model on $VB(N, \mathfrak{A})$. Let M be a Banach manifold of class C^{r+s}, $s \geq 3$, modelled on Banach spaces in $\mathfrak{A}$ and admitting a connection of class C^{r+s-2}. M is furthermore to be without boundary.

Theorem 5.1. *Under the assumptions above, there exists a unique Banach manifold* $\mathfrak{S}(N, M)$ *of class* C^{s-2}, *such that if* $\exp : \mathcal{O} \to M$, $\mathcal{O} \subset TM$, *is the exponential map corresponding to any* C^{r+s-2} *connection on* M, $\mathcal{D} \subset \mathcal{O}$ *an open neighborhood of the set of zero vectors in* TM *such that* $(\tau, exp) | \mathcal{D}$ *is a diffeomorphism* (§3), h: $N \to M$ *a map of class* C^r. *Then*

$$\mathfrak{S}(\exp) : \mathfrak{S}(h^* \mathcal{D}) \to \mathfrak{S}(N, M) \ by \ \xi \mapsto \exp \circ \xi ,$$

and is chart for $\mathfrak{S}(N, M)$, called a natural chart centered at h.

Proof. Let $E_h = h^*TM$ be the pull-back of $\tau : TM \to M$ by h and $\mathscr{D}_h = h^*\mathscr{D} \subset E_h$. Then E_h is in $VB(N, \mathfrak{A})$ and $\mathscr{D}_h$ is an open neighborhood of the set of zero vectors in E_h, such that with $\pi_h = h^*\tau$:

$$\Phi_h = (\pi_h, \exp) : \mathscr{D}_h \to N \times M$$

is a diffeomorphim of class C^r onto an open neighborhood U_h of the graf of h in $N \times M$. We define $\mathfrak{S}(N, M)$ to be the set of maps $g \in C^0(N, M)$, such that there is an $h \in C^r(N, M)$ with graf $(g) \subset U_h$ and $\Phi_h^{-1} \circ (id, g) \in \mathfrak{S}(\mathscr{D}_h)$. We then define $\mathfrak{S}(U_h)$ to be the set of the g's with this property and

$$\mathfrak{S}(\Phi_h^{-1}) : \mathfrak{S}(U_h) \to \mathfrak{S}(\mathscr{D}_h) \text{ by } \mathfrak{S}(\Phi_h^{-1})(g) = \Phi_h^{-1} \circ (id, g) .$$

This is a one-to-one mapping with

$$\mathfrak{S}(\Phi_h^{-1})^{-1} = \mathfrak{S}(\Phi_h) : \xi \longmapsto \exp \circ \xi .$$

Let $f \in C^r(N, M)$ be another map, such that $\mathfrak{S}(U_h) \cap \mathfrak{S}(U_f)$ is not empty. Then $U_h \cap U_f$ contains the graf of a continuous map, so $\mathscr{O}_h = \Phi_h^{-1}(U_h \cap U_f)$ is an open subset of E_h projected onto N. Now

$$\Phi_{hf} = \Phi_f^{-1} \circ \Phi_h : \mathscr{O}_h \to E_f$$

is a fibre preserving map of class (C^r, C^{r+s-2}), therefore

$$\mathfrak{S}(\Phi_f^{-1}) \circ \mathfrak{S}(\Phi_h) = \mathfrak{S}(\Phi_{hf}) : \mathfrak{S}(\mathscr{O}_h) \to \mathfrak{S}(E_f)$$

and is of class C^{s-2} by Lemma 4.1. Hence the collection $(\mathfrak{S}(U_h), \mathfrak{S}(\Phi_h^{-1}))$, $h \in C^r(N, M)$ is an atlas of class C^{s-2} for $\mathfrak{S}(N, M)$ and defines a topology and differentiable structure of class C^{s-2} on $\mathfrak{S}(N, M)$ the topology is easily seen to be Hausdorff. The differentiable structure does obviously not depend on the connection on M used, so long as it is of class C^{r+s-2}.

Remark. In particular, the Theorem holds for $\mathfrak{S} = C^k$, $0 \leq k \leq r$, $k < \infty$. Note, that if $\mathfrak{S}(E)$ is defined to be the closed subspace of $C^k(E)$ vanishing on the boundary of N (if not empty) for all $E \in VB(N, \mathfrak{A})$, then $\mathfrak{S}(N, M)$ consists of all maps $f \in C^k$ (N, M) such that $f \mid \partial N \in C^r(\partial N, M)$ and is rather disconnected.

Theorem 5.2. *With $s \geq 4$ and else the same assumptions as in Theorem 5.1, we have a vector bundle:*

$$\mathfrak{S}(\tau) : \mathfrak{S}(N, TM) \to \mathfrak{S}(N, M), \ \mathfrak{S}(\tau)(\eta) = \tau \circ \eta$$

of class C^{s-3}, which is naturally equivalent to the tangent bundle of $\mathfrak{S}(N, M)$. Moreover given any connection on M, let $\mathfrak{S}(\exp) : \mathfrak{S}(\mathscr{D}_h) \to \mathfrak{S}(N, M)$ be the natural chart centered at $h \in C^r(N, M)$. Then

$$\mathfrak{S}(\nabla_2 \exp) : \mathfrak{S}(\mathscr{D}_h) \times \mathfrak{S}(E_h) \to \mathfrak{S}(N, TM)$$
$$\mathfrak{S}(\nabla_2 \exp)(\xi, \eta) = \nabla_2 \exp \circ (\xi, \eta)$$

gives a trivialization of $\mathfrak{S}(\tau)$ *over* $\mathfrak{S}(\exp)$ *corresponding to the tangent trivialization* $T\mathfrak{S}(\exp)$ *under the bundle equivalence.*

Proof. Given a C^{r+s-2}-connection on M we have a C^{r+s-3} connection on TM by Theorem 3.1., therefore $\mathfrak{S}(N, TM)$ is a Banach manifold of class C^{s-3}. We will use the natural atlas for the manifold $\mathfrak{S}(N, TM)$ constructed by using the induced connection on TM with exponential map $\exp_T$.

We define $0h : N \to TM$ as h followed by the zero section of τ, then $0h$ is of class C^r as well as h. We define a bundle equivalence $I_h : E_h \oplus E_h \to E_{0h} = (0h)^*T^2M$, by

$$I_h(\xi_p, \eta_p) = (T\tau, \tau_1, K)^{-1}(\xi_p, 0, \eta_p)$$

where K is the connection map for TM (§3). Then with $\mathscr{D}_{0h} = I_h(\mathscr{D}_h \oplus E_h)$, $\Phi_{0h} = (\pi_{0h}, \exp_T) : \mathscr{D}_{0h} \to N \times TM$ is a diffeomorphism of class C^{r+s-3} and

$$[\Phi_{0h} \circ I_h](\xi_p, \eta_p) = (p, \nabla_2 \exp(\xi_p, \eta_p))$$

using Corollary 3.1. $\mathfrak{S}(\Phi_{0h}^{-1})$ is then a chart for $\mathfrak{S}(N, TM)$ by definition. Consider the diagram

$$\begin{array}{ccccccc}
T\mathfrak{S}(U_h) & \xrightarrow{T\mathfrak{S}(\Phi_h^{-1})} & \mathfrak{S}(\mathscr{D}_h) \times \mathfrak{S}(E_h) & \xrightarrow{\mathfrak{S}(I_h)} & \mathfrak{S}(\mathscr{D})_{0h} & \xrightarrow{\mathfrak{S}(\Phi_{0h})} & \mathfrak{S}(U_{0h}) \\
\downarrow & & \downarrow & & & & \downarrow \\
\mathfrak{S}(U_h) & \xrightarrow{\mathfrak{S}(\Phi_h^{-1})} & \mathfrak{S}(\mathscr{D}_h) & & \xrightarrow{\mathfrak{S}(\Phi_h)} & & \mathfrak{S}(U_h)
\end{array}$$

We claim $\mathfrak{S}(T) = \mathfrak{S}(\Phi_{0h} \circ I_h) \circ T\mathfrak{S}(\Phi_h^{-1})$ does not depend on h and is a diffeomorphism of class C^{s-3}. We have $T\mathfrak{S}(\Phi_{hf})(\xi, \eta) = (\mathfrak{S}(\Phi_{hf})(\xi), \mathfrak{S}(D_2\Phi_{hf})(\xi) \cdot \eta)$ using Lemma 4.1. Now $\exp(\Phi_{hf}(\xi_p)) = \exp(\xi_p)$ so

$$\nabla_2 \exp(\Phi_{hf}(\xi_p), D_2\Phi_{hf}(\xi_p) \cdot \eta_p) = \nabla_2 \exp(\xi_p, \eta_p)$$

and then by the above formula for $\Phi_{0h} \circ I_h$ we are done. $\mathfrak{S}(T)$ is of class C^{s-3} as $\mathfrak{S}(I_h)$ is. This proves that $\mathfrak{S}(\tau)$ is a vector bundle equivalent to the tangent bundle as the above diagram is commutative and linear on fibres, where the linear space structure of $\mathfrak{S}(\tau)^{-1}(g)$ is of course the one inherited from sections in g^*TM, which is a bundle of class C^0 at least. This shows that we can extend $\mathfrak{S}$ to all bundles g^*TM for $g \in \mathfrak{S}(N, M)$ and the above map $\mathfrak{S}(T)$ is a toplinear isomorphism of the tangent space over g with $\mathfrak{S}(g^*TM)$ so the bundle equivalence is natural. q.e.d.

We will in the following use I_h and $\mathfrak{S}(T)$ as identification maps. Note that it was not a priori clear whether $\mathfrak{S}(h^*TM)$ coincides with the set of maps $\xi \in \mathfrak{S}(N, TM)$ s.t. $\tau \circ \xi = h$.

Theorem 5.3. *Let M, M' be Banach manifolds of class C^{r+s}, $s \geq 4$, and else the same situation as in Theorem 5.1. Let $\theta : M \to M'$ be a map of class C^{r+s-2}. Then $\mathfrak{S}(\theta) : \mathfrak{S}(N, M) \to \mathfrak{S}(N, M')$, $g \to \theta \circ g$, is of class C^{s-2} and $T\mathfrak{S}(\theta) = \mathfrak{S}(T\theta)$.*

Here we have identified the tangent bundles with $\mathfrak{S}(N, T^r M)$, $T^r\mathfrak{S}(\theta) = \mathfrak{S}(T^r\theta)$, $r \leq s-2$, follows then by induction.

Proof. Let $h \in C^r(N, M)$, and $f \in C^r(N, M)$ s.t. $\theta \circ h \in \mathfrak{S}(U_f)$. Then $\theta_{hf} = \Phi_f^{-1} \circ (id, \theta) \circ \Phi_h : \mathcal{D}'_h \to E_f$ is a fibre preserving map of class (C^r, C^{r+s-2}) and $\mathfrak{S}(\theta_{hf})$ is the local representative of θ. Now by Lemma 4.1 $\mathfrak{S}(\theta_{hf})$ is of class C^{s-2} and $D\mathfrak{S}(\theta_{hf}) = \mathfrak{S}(D_2\theta_{hf})$. Moreover we have $\exp' \circ \theta_{hf} = \theta \circ \exp$, so

$$\nabla_2 \exp' \circ T_2\theta_{hf} = T\theta \circ \nabla_2 \exp, \text{ with } T_2\theta_{hf}(\xi, \eta) = (\theta_{hf}(\xi), D_2\theta_{hf}(\xi) \cdot \eta)$$

which shows in the light of Theorem 5.2 that $\mathfrak{S}(T_2\theta_{hf})$ is the local tangent of $\mathfrak{S}(\theta_{hf})$.

Theorem 5.4. *Let K be a connection map for the manifold M in Theorem 5.1. Then $\mathfrak{S}(K) : A \mapsto K \circ A$, is a connection map for $\mathfrak{S}(N, M)$, the connection is of class C^{s-4} and the local connector is given by*

$$\mathfrak{S}(\Lambda_h) : \mathfrak{S}(\mathcal{D}_h) \to L^2(\mathfrak{S}(E_h), \mathfrak{S}(E_h)) \text{ with } \Lambda_h = h^*\Lambda \quad (\S 3).$$

The corresponding exponential map is just $\mathfrak{S}(\exp)$.

Proof. We have only to prove the local formula, as $\Lambda_h : \mathcal{D}_h \to L^2(E_h, E_h)$ is of class (C^r, C^{r+s-4}) and we have a continuous linear inclusion $\mathfrak{S}(L^2(E_h, E_h)) \subset L^2(\mathfrak{S}(E_h), \mathfrak{S}(E_h))$ applying the manifold model property 2) for $\mathfrak{S}$ twice. So let $\xi \in \mathfrak{S}(\mathcal{D}_h)$ and $\eta, \zeta_1, \zeta_2 \in \mathfrak{S}(E_h)$ be the local components of some $A \in T^2\mathfrak{S}(N, M) = \mathfrak{S}(N, T^2M)$. Then

$$\begin{aligned} \tau_1 \circ A &= \nabla_2 \exp \circ (\xi, \eta), \\ K \circ A &= \nabla_2 \exp \circ (\xi, \zeta_2) + \nabla_2\nabla_2 \exp \circ (\xi, \eta, \zeta_1) \\ &= \nabla_2 \exp (\xi, \zeta_2 + (\Lambda \circ \xi) \cdot (\eta, \zeta_1)), \end{aligned}$$

as

$$\begin{aligned} &K\left(\frac{d}{dt}\nabla_2 \exp(\xi(p) + t\zeta_1(p), \eta(p) + t\zeta_2(p)) \,|\, t = 0\right) \\ &\quad = \nabla\nabla_2 \exp (\xi(p), \eta(p), 0, \zeta_1(p), \zeta_2(p)) \end{aligned}$$

and then using Corollary 3.2 b) and the definition of Λ in §3. A somewhat longer but more precise proof can be obtained by using $\nabla_2 \exp_T$ and computing the connection with exp from Corollary 3.1 and the defining properties of the connection map K_T. By Lemma 3.1 b) we have $\mathfrak{S}(\Lambda_h)(t\xi) \cdot (\xi, \xi) = 0$, which proves that $t\xi$, $0 \leq t \leq 1$, is a geodesic in this chart. This means that $\mathfrak{S}(\exp) : \xi \mapsto \exp \circ \xi$ is the exponential map. The natural charts are exactly the "normal coordinates."

6. Bundles of sections over manifolds of maps

Let N, $\mathfrak{A} \subset \mathfrak{B}$, $\mathfrak{S}$ and M be given as in §5. Suppose we have a functor $\lambda : \mathfrak{A} \to \mathfrak{A}$ of class C^{r+s-3} and covariant lets say. The corresponding map of morphisms:

$$\lambda_* : L(E, F) \to L(\lambda(E), \lambda(F))$$

is then of class C^{r+s-3} and we have an induced functor

$$\lambda_N : VB(N, \mathfrak{A}) \to VB(N, \mathfrak{A})$$

with $\lambda_N(E)_p = \lambda(E_p)$, $p \in N$ (see Lang [6]) and the fibre preserving map λ_{*N} (given by λ_* on each fibre) is of class (C^r, C^{r+s-3}). Moreover the map of morphisms for λ_N is

$$\lambda_{N*} = C^r(\lambda_{*N}) : (A, B) \longmapsto \lambda_{*N} \circ (A, B) .$$

Theorem 6.1. *Let N, $\mathfrak{A}$, $\mathfrak{S}$, M, λ be as above and let $\mathfrak{T}$ be a section functor on $VB(N, \mathfrak{A})$, such that $\mathfrak{S}L \subset L(\mathfrak{T}, \mathfrak{T})$ holds* (§4). *Then $\mathfrak{T}$ can be uniquely extended over the vector bundles $\lambda_N(f^*TM)$ for $f \in \mathfrak{S}(N, M)$ and the union of all the Banach spaces $\mathfrak{T}(\lambda_N(f^*TM))$ is a vector bundle $\mathfrak{T}(\lambda_N(\mathfrak{S}(N, M)^*TM))$ of class C^{s-3} over $\mathfrak{S}(N, M)$.*

Proof. For any continuous maps $\alpha, \beta : N \to M$, with $\beta \in C^0(U_\alpha)$ say $\beta = \exp \circ \xi$, $\xi \in C^0(\mathscr{D}_\alpha) \subset C^0(\alpha^*TM)$ we will define a section:

$$J_{\alpha\beta} \in C^0(L(\alpha^*TM, \beta^*TM))$$

by $J_{\alpha\beta}(p) \cdot \eta_p = \nabla_2 \exp(\xi(p), \eta_p)$. Then $J_{\alpha\beta}$ is a toplinear isomorphism on each fibre. Now let $h, f \in C^r(N, M)$ with $\mathfrak{S}(U_h) \cap \mathfrak{S}(U_f) \neq \phi$. Then with $\mathcal{O}_h = \Phi_h^{-1}(U_h \cap U_f)$ we have

$$D_2\Phi_{hf} : \mathcal{O}_h \to L(E_h, E_f)$$

and this is a fibre preserving map of class (C^r, C^{r+s-3}), so $\mathfrak{S}(D_2\Phi_{hf})$ is of class C^{s-3} by Lemma 4.1. Moreover

$$\mathfrak{S}(D_2\Phi_{hf})(\xi) = (J_{fg})^{-1} \cdot J_{hg}, \; g = \exp \circ \xi$$

as $\nabla_2 \exp \circ (\Phi_{hf} \circ \xi, (D_2\Phi_{hf} \circ \xi) \cdot \eta) = \nabla_2 \exp \circ (\xi, \eta)$, for any $\xi \in \mathfrak{S}(\mathcal{O}_h)$. We define

$$\mathfrak{T}(\lambda_N(g^*TM)) = \{(\lambda_{*N} \circ J_{hg}) \cdot \eta : \eta \in \mathfrak{T}(\lambda_N(E_h))\} .$$

This definition does indeed not depend on h as

$$(\lambda_{*N} \circ J_{fg}) \cdot (\lambda_{*N} \circ (D_2\Phi_{hf} \circ \xi)) = \lambda_{*N} \circ J_{hg}$$

and

$$\lambda_{*N} : L(E_h, E_f) \to L(\lambda_N(E_h), \lambda_N(E_f))$$

is of class (C^r, C^{r+s-3}), $s - 3 \geq 0$, so

$$\lambda_* \circ (D_2\Phi_{hf} \circ \xi) \in \mathfrak{S}(L(\lambda_N(E_h), \lambda_N(E_f)))$$

which in turn is continuous linearly included in

$$L(\mathfrak{T}(\lambda_N(E_h)), \mathfrak{T}(\lambda_N(E_f))$$

by property $\mathfrak{S}L \subset L(\mathfrak{T}, \mathfrak{T})$. The local trivialization over the chart $(\mathfrak{S}(U_h), \mathfrak{S}(\Phi_h^{-1}))$ of $\mathfrak{S}(N, M)$ is now given by the linear isomorphism $\lambda_{*N} \circ J_{hg}^{-1}$ in the fibre over g and defines a Banach space topology on each fibre. The transition map between charts centered at h and f, is the composite of

$$\begin{aligned} &\mathfrak{S}(\mathcal{O}) \to \mathfrak{S}(L(h^*TM, f^*TM)) \to \\ &\to \mathfrak{S}(L(\lambda_N(h^*TM), \lambda_N(f^*TM))) \to \\ &\to L(\mathfrak{T}(\lambda_N(h^*TM)), \mathfrak{T}(\lambda_N f^*TM))) , \end{aligned}$$

where the first map $\mathfrak{S}(D_2\Phi_{hf})$ and the second $\mathfrak{S}(\lambda_{*N})$ are of class C^{s-3} and the last one is a continuous linear inclusion. q.e.d.

The theorem holds in particular for $\mathfrak{T} = \mathfrak{S}$. Observe that $\lambda_N(f^*TM) = f^*\lambda_M(TM)$, so the fibre over f consists of sections in the pull-back of the tensor bundle of type λ over M by f, t.i. tensor fields of type λ along the map f. For $\lambda = id$, we get exactly the tangent bundle of $\mathfrak{S}(N, M)$, the local trivialization used here is moreover the same as used for $\mathfrak{S}(N, TM)$ before. If X is a tensor field of type λ on M, t.i. a section in $\lambda_M(TM)$, we have an induced section $\mathfrak{S}^*X$ in $\mathfrak{S}(\lambda_N(\mathfrak{S}(N, M)^*TM))$ defined by $(\mathfrak{S}^*X)(f) = f^*X = X \circ f$, which is of class C^{s-3} if X is of class C^{r+s-3}.

We note that λ could just as well have been contravariant or of mixed variance. Moreover if λ is of several variables, we could replace some by fixed bundles from $VB(N, \mathfrak{A})$ and the rest by pull-backs of TM by maps in $\mathfrak{S}(N, M)$. However, as TN is not in $VB(N, \mathfrak{A})$ if $r < \infty$, we will have to make some additional assumptions before applying the theorem.

Let $r \geq 1$ and denote by $VB^i(N, \mathfrak{A})$ the category of vector bundles of class C^i, $0 \leq i \leq r$, over N with fibres in $\mathfrak{A}$. Section functors and manifold models on $VB^i(N, \mathfrak{A})$ are defined just as before (replacing r by i). We will hanceforth suppose $\mathfrak{S}$ is a manifold model on $VB^{r-1}(N, \mathfrak{A})$. Let $\lambda : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ be a functor of class C^{r+s-4} and $\mathfrak{T}$ a section functor on $VB^{r-1}(N, \mathfrak{A})$ with $\mathfrak{S}L \subset L(\mathfrak{T}, \mathfrak{T})$. Then $\mathfrak{T}(\lambda_N(TN, \mathfrak{S}(N, M)^*TM))$ is a vector bundle of class C^{s-3} over $\mathfrak{S}(N, M)$ with $\mathfrak{T}(\lambda_N(TN, f^*TM))$ as the fibre over f. The proof is exactly the same as in previous theorem, replacing λ_{*N} by $\lambda_{*N}(I, \cdot)$, where I is the identity section in $L(TN, TN)$.

Let $\mathfrak{T}_1, \mathfrak{T}_2$ be section functors in $VB^i(N, \mathfrak{A})$ and $VB^{i-1}(N, \mathfrak{A})$ respectively. We will call $\mathfrak{T}_1$ of higher degree than $\mathfrak{T}_2$, if the following holds: Given $E \in VB^i(N, \mathfrak{A})$ and any connection of class C^{i-1} on E, then we have a unique extension of the covariant derivative to a continuous linear map

$$\nabla : \mathfrak{T}_1(E) \to \mathfrak{T}_2(L(TN, E)) .$$

This means that $C^i(E) \cap \mathfrak{T}_1(E)$ is dense in $\mathfrak{T}_1(E)$ and $\nabla : C^i(E) \cap \mathfrak{T}_1(E) \to C^{i-1}(L(TN, E)) \cap \mathfrak{T}_2(L(TN, E))$ is continuous, in the $\mathfrak{T}_1, \mathfrak{T}_2$ topologies. Obviously C^k is of higher degree than C^j for $k > j$. We will say that $\mathfrak{T}_1$ is contained in $\mathfrak{T}_2$, if we have a continuous linear inclusion $\mathfrak{T}_1(E) \subset \mathfrak{T}_2(E)$ for every vector bundle E on which both $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are defined.

Theorem 6.2. *Let N be a compact manifold of class C^r, $r \geq 1$, $\mathfrak{S}$ a manifold model on $VB^{r-1}(N, \mathfrak{A})$ and M a Banach manifold of class C^{r+s}, $s \geq 3$, as before. Let $\mathfrak{T}$ be a section functor on $VB^{r-1}(N, \mathfrak{A})$, such that $\mathfrak{S}L \subset L(\mathfrak{T}, \mathfrak{T})$, $\mathfrak{S}$ is of higher degree than $\mathfrak{T}$ and contained in $\mathfrak{T}$. Then $\partial : f \mapsto \partial f \in C^{r-1}(L(TN, f^*TM))$ for $f \in C^r(N, M)$ can be extended uniquely to a section ∂ of class C^{s-3} in $\mathfrak{T}(L(TN, \mathfrak{S}(N, M)^*TM))$. Moreover in the local trivialization centered at $h \in C^r(N, M)$ constructed above, the local representative ∂_h is given by*

$$\partial h = \mathfrak{S}(\vartheta_h) + \nabla_h : \mathfrak{S}(\mathscr{D}_h) \to \mathfrak{T}(L(TN, h^*TM)) ,$$

where $\nabla_h = h^*\nabla$ is the pull-back of the covariant derivation on M obtained from the connection used to construct the natural chart at h and

$$\vartheta_h = (h^*\vartheta) \cdot \partial h, \text{ where } \vartheta : \mathscr{D} \to L(TM, TM)$$

is the twist map from §3. Explicitly,

$$\nabla_2 \exp(\xi_p, \vartheta_h(\xi_p) \cdot v_p) = \nabla_1 \exp(\xi_p, \partial h(p) \cdot v_p) .$$

Proof. ∇_h is a covariant differentiation coming from a connection on E_h of class C^{r-1}, and has therefore by the assumption that $\mathfrak{S}$ is of higher degree than $\mathfrak{T}$ a unique continuous linear extension to

$$\nabla_h : \mathfrak{S}(E_h) \to \mathfrak{T}(L(TN, E_h)) .$$

Moreover ϑ_h is a fibre preserving map of class (C^r, C^{r+s-3}) and thus by Lemma 4.1 induces a C^{s-3}-map:

$$\mathfrak{S}(\vartheta_h) : \mathfrak{S}(\mathscr{D}_h) \to \mathfrak{S}(L(TN, E_h)) \subset \mathfrak{T}(L(TN, E_h))$$

where the inclusion is continuous linear by the assumption that $\mathfrak{S}$ is contained in $\mathfrak{T}$. We have therefore only to prove that the local formula is correct for sections of class C^r in $\mathfrak{S}(\mathscr{D}_h)$ as those are dense. So let $f = \exp \circ \xi$ with $\xi \in \mathfrak{S}(\mathscr{D}_h)$ and of class C^r; then

$$\begin{aligned} \partial f(p) \cdot v &= [\nabla \exp \circ (\pi_1, T\pi, K) \circ T\xi](v) \\ &= \nabla_1 \exp(\xi(p), \partial h(p) \cdot v) + \nabla_2 \exp(\xi(p), \nabla_h \xi(p) \cdot v) \\ &= J_{hf}(p) \cdot (\vartheta_h(\xi(p) \cdot v + \nabla_h \xi(p) \cdot v) \end{aligned}$$

which proves the local formula.

Theorem 6.3. *Let the assumption be as in previous theorem and give M a connection of class C^{r+s-2}. Then the pull-back of the covariant derivation on M has a unique extension to a vector bundle map*

$$\nabla^* : \mathfrak{S}(\mathfrak{S}(N, M)^* TM) \to \mathfrak{T}(L(TN, \mathfrak{S}(N, M)^* TM))$$

of class C^{s-4} provided $s \geq 4$. Moreover in the natural local trivialization centered at $h \in C^r(N, M)$ the local representative of ∇^*:

$$\nabla_h^* : \mathfrak{S}(\mathscr{D}_h) \to L(\mathfrak{S}(E_h), \mathfrak{T}(L(TN, E_h)))$$

is given by the formula

$$\nabla_h^*(\xi) = \nabla_h + \mathfrak{S}(D_2\vartheta_h)(\xi) + \mathfrak{S}(\Lambda_h)(\xi) \cdot \partial_h \xi ,$$

where $\nabla_h = h^*\nabla$, ϑ_h and ∂_h are as in Theorem 6.2 and Λ_h is the local connector from Theorem 5.4.

Proof. $D_2\vartheta_h$ and Λ_h are both fibre preserving maps of class (C^r, C^{r+s-4}) from $\mathscr{D}_h$ into $L(E_h, L(TN, E_h))$ and $L^2(E_h, E_h) \to L(L(TN, E_h), L(E_h, L(TN, E_h)))$ respectively, so ∇_h^* is of class C^{s-4} and we have only to prove the local formula under the additional hypothesis that ξ is of class C^r. Let $f = \exp \circ \xi$, $\zeta = J_{hf} \cdot \eta = \nabla_2 \exp(\xi, \eta)$ with $\eta \in C^r(E_h)$. Then

$$\begin{aligned} \nabla_f \zeta &= \nabla\nabla_2 \exp(\xi, \eta, \partial h, \nabla\xi, \nabla\eta) \\ &= \nabla_1\nabla_2 \exp(\xi, \eta, \partial h) + \nabla_2\nabla_2 \exp(\xi, \eta, \nabla\xi) + \nabla_2 \exp(\xi, \nabla\eta) \\ &= \nabla_2 \exp(\xi, \nabla\eta + (D_2\vartheta \circ \xi) \cdot (\eta, \partial h) + (\Lambda \circ \xi) \cdot ((\vartheta \circ \xi) \cdot \partial h + \nabla\xi, \eta)) \end{aligned}$$

by Lemma 3.1 c) and definition of Λ. The local formula follows as we have $D_2\vartheta_h(\xi_p) \cdot \eta_p = D_2\vartheta(\xi_p) \cdot (\eta_p, \partial h(p))$.

Theorem 6.4. *Given N, $\mathfrak{A}$, $\mathfrak{S}$, M as in Theorem 6.3 with $r \geq 2$, $s \geq 4$ and let both N and M have connections of class C^{r-2} and C^{r+s-2} respectively. Let $\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3$ be section functors on $VB^{r-1}(N, \mathfrak{A})$ and suppose $\mathfrak{S}L \subset L(\mathfrak{T}_i, \mathfrak{T}_i)$ for $i = 1, 2, 3$; $\mathfrak{T}_1 L \subset L(\mathfrak{T}_2, \mathfrak{T}_3)$; $\mathfrak{S}$ is of higher degree and contained in $\mathfrak{T}_1$ and $\mathfrak{T}_2$ is of higher degree than $\mathfrak{T}_3$. Then taking the induced connection for the bundles $L^k(TN, f^*TM)$, $f \in C^r(N, M)$ (see §2), there is a unique extension of the covariant derivation to a vector bundle map*

$$\nabla^* : \mathfrak{L}_2(L^k(TN, \mathfrak{S}^*TM)) \to \mathfrak{L}_3(L^{k+1}(TN, \mathfrak{S}^*TM))$$

of class C^{s-4} *for* $k \geq 0$. *Moreover we have exactly the same local formula with the obvious interpretation of* ϑ_h *and* Λ_h.

Proof. It is an easy matter to check the local formula. We use the inclusion:

$$L^2(E_h, E_h) \to L(L(TN, E_h), L(L^k(TN, E_h), L^{k+1}(TN, E_h))$$

to follow Λ_h and similar inclusions to follow ϑ_h and $D_2\vartheta_h$ to obtain the correct maps. Our assumptions are easily seen to ensure that ∇^* is of class C^{s-4}.

Corollaries for C^k. *Let N be a compact Riemannian manifold of class* C^∞ *and M a paracompact Banach manifold of class* C^∞, *without boundary and with a* C^∞*-connection and a Finsler structure. Then for* $0 \leq k < \infty$:

1. $C^k(N, M)$ *is a paracompact Banach manifold of class* C^∞, *with* $C^k(N, TM)$ *as a tangent bundle space and admits a* C^∞*-connection and a Finsler structure.*

2. The connection on M induces a canonical C^∞*-connection on* $C^k(N, M)$, *such that* $C^k(\exp)$ *is the exponential map for* $C^k(N, M)$, *if* exp *denotes the exponential map for M.*

3. $C^s(L^r(TN, C^k(N, M)^*TM)) \to C^k(N, M)$ *is a vector bundle of class* C^∞ *for* $0 \leq s \leq k, 0 \leq r < \infty$.

4. The tangent derivative $\partial : C^k(N, M) \to C^{k-1}(L(TN, C^k(N, M)^*TM))$ *is a* C^∞ *section in this bundle.*

5. The connections on N and M induce C^∞ *sections in*

$$L(C^s(L^r(TN, C^k(N, M)^*TM)), C^{s-1}(L^{r+1}(TN, C^k(N, M)^*TM)))$$

for $r \geq 0$, *which are given by the covariant derivative on each fibre defined as in* §2, *taking the induced connection for the pull-backs.*

6. We have an induced Finsler structure on $C^k(N, M)$ *given on the fibre* $C^k(f^*TM)$, $f \in C^k(N, M)$, *by*

$$\|\xi\|_{C^k} = \sum_{i=0}^{k} \sup_{p \in N} \|\nabla^i \xi(p)\|,$$

where the Riemannian metric on N and the pull-back of the Finsler structure from M are used to obtain a Finsler structure for $L^i(TN, f^*TM)$, $i \geq 0$.

Remark. If M is finite dimensional Riemannian manifold and $k > \frac{1}{2} \dim N$, we obtain the same results for the Sobolev chain H^s using the properties of H^s as a section functor (see [5]), except we replace the Finsler structure by the Riemannian metric

$$<\xi, \eta>_k = \sum_{i=0}^{k} \int_N <\nabla^i \xi, \nabla^i \eta>$$

where we integrate with respect to the Riemannian measure on N and the Riemannian metrics for $L^i(TN, f^*TM)$ are constructed from the Riemannian metrics for TN and TM. A k-th order energy function $E_k : H^k(N, M) \to R$ may be defined by

$$E_k(f) = \frac{1}{2} \sum_{i=0}^{k-1} \int_N \| \nabla^i \partial f \|^2$$

and is of class C^∞ as is easily seen applying theorems 6.2-4. E_k satisfies condition C.

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